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Polytope Typology D: The All-space-filling Periodic Honeycombs Separation and Progression of Polyhedra

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Abstract

In previous research, I classified the 3D polyhedra and the 2D tessellations of the plane according to class dependent on their symmetry, and organized them into a 3-frequency polar zonahedral schema, while identifying the constituent sequences through which the 2D and 3D polytopes might be considered to evolve from simple VP to complex GR, in combination with a rhombic schema for their progression of faces. Here, I further develop and clarify the order of the periodic all-space-filling polyhedral arrays (the honeycombs) by addressing the separation and progression of the polyhedra in the honeycombs to better comprehend their overall morphology.

Keywords: all-space-filling periodic arrays, honeycombs, separation of polyhedra, classification, order

1. Introduction

My classification of the polyhedra [1−7] includes in each class 8 Primary Polyhedra (PP): the Verticial Polytope (VP) simplex, +ve & −ve Polar ($PL^{+/-}$), QuasiRegular (QR), +ve & −ve truncated polar ($TP^{+/-}$), Small Rhombic QR (SR), and Great Rhombic QR (GR). The $3f$ polar zonahedral (PZ_3) schema of the polyhedra generalizes their faces to encompass the surface polytopes (PTs) of which they are composed, so a polyhedron can be considered in terms of its vertices (VTs), edges (EGs), and faces (Fs). Certain of these VTs and EGs are principial—those oriented orthogonal to and coaxial with the axes of symmetry; the rest are incidental, like the VTs and EGs of the SQs of the $SRCO$, and of the hexagons (HXs) and octagons (OGs) of the $GRCO$, respectively; any PP can be considered to consist of its principial +ve, neutral (ntrl), or −ve surface PTs. In the evolution of form $VP \rightarrow GR$ in six paths (e.g., $VP \rightarrow CB \rightarrow SR \rightarrow GR$) [6: fig. 5], each PT develops according to the progression of faces (PoF) schema. Separation of faces (SoF) of these PPs uniquely determines their PZ_3 schema morphology.

In this paper, I develop that research by applying the polyhedral SoF and PoF to further investigate the sequences of polyhedra in the honeycombs, by considering the analogous Separation of Polyhedra (SoP), and Progression of Polyhedra (PoP), comparing their morphology with that of the polyhedra. This paper, the fourth in a series, is best read in conjunction with a companion paper on the Class III honeycombs that is well illustrated [8], both being available for download as color PDFs.

Overview

The honeycombs consist of polytopes (PTs) that can first be classified into Primary elements that form regular arrays, and the Neutral polyTopes (NTs) that mediate them, then by class according to their formal spatial distribution. Most constitute the regular and semi-regular polyhedra; but in my analysis they also include VPs, and as NTs, the regular Octagonal Prism (OP); and several virtual elements that include the Neutral Vertex (NX), Neutral Square (NS), Neutral Rhomb (NR; rotated SQ), and Neutral Octagon (NO), also the Neutral Edge (NE); together with their respective projected forms, the Vertex Prism (XP), Square Prism (SP), Rhombic Prism (RP; rotated SP), and Octagonal Prism (OP), also the Edge Prism (EP); these are considered the base PTs, and those formed by their prismatic projection in the various SoPs, respectively [8: table II]. NTs have a unique main axis that corresponds to one of the XYZ axes, as well as minor axes, while the Primary elements have regular polyaxial symmetry, corresponding to the 8 members of the Class II of the Polyhedra of {2,3,4} symmetry; and the simplex

Verticial Polytope (VP_I) and the polar and truncated tetrahedral (T) members of Class I of {2,3,3} symmetry of the 2 polar tetrahedra ($TH^{+/-}$), and 2 truncated tetrahedra ($TT^{+/-}$), where the polarity of the TT is that of the TH that it is truncated from. Class II of the Polyhedra consists of the Verticial Polytope (VP_{II}) simplex, Octahedron (OH), Cube (CB), Truncated Octahedron (TO), Truncated Cube (TC), Cuboctahedron (CO), Small Rhombic Cuboctahedron ($SRCO$), and Great Rhombic Cuboctahedron ($GRCO$). All elements have unit edge length, except the simplex VP and its virtual $VT^{+/0/-}$s. To trace the evolution of form from honeycomb to related honeycomb by the SoP in Classes II and III of the Honeycombs, I thus include certain VTs, EGs, and Fs as NTs, and the 3D prism (OP).

Figure 1 shows that the PPs in the honeycombs are then located centered on the nodes of two primary cubic lattices (PCLs; in Class III), or on the nodes of their respective pair of primary tetrahedral lattices (PTLs; in Classes I and II). In general, the neutral elements of any honeycomb are of two kinds (though if P_1=P_2, these are the same), have a major axis parallel to one of the XYZ axes (so appear in three orientations according to their XYZ axis), and are coaxial to the edges of the reference cubic lattices, but are *not* in general located at the mid-point of their edges, except where the two PPs they mediate are of the same kind. According to the spatial pattern of the PPs on the cubic or tetrahedral lattices, just three classes can be identified: Class I has a solitary honeycomb of +ve and −ve THs and TTs (although in principle, the 4 Class III honeycombs [GR×GR], [TO×TO], [CB×CB], and [VP×VP] where PP_1 = PP_2, might, if colored, also be included). Class II has four honeycombs, two of which include the +ve and −ve THs, the other two including the +ve and −ve TTs. Class III by my classification consists of 4 clusters of 4 honeycombs (Fig. 2), with two clusters being reflections of one another, and 2 clusters being self-reflective, so number 10 kinds in all, including the simplex [VP×VP] that consists of a single point. By my schema, things need not be what they appear: e.g., the common lattice of cubes could be a [CB×VP], [VP×CB], or [CB×CB] honeycomb, depending on the coloring of the CBs. My classification is, to the best of my ability, rigorous, and strongly related to my treatment of the (regular and semiregular) polyyhedra, as elsewhere exhaustively detailed. These explorations suggest a formal schema to represent the morphology of the honeycombs that I intend to later develop.

The formal structure of the three classes of honeycombs

The locations of the various PTs vary according to the class of their honeycomb, and their status as +ve or −ve PPs that are in juxtaposition to one another, or the Neutral PTs (NTs) that mediate them. Consider two primary cubic lattices (PCLs), where the nodes of one PCL lie at the centers of the cubes of the other PCL and *vice versa* (Fig. 1). The two can be designated PCL_1 and PCL_2, *wlog*. Either PCL can be differentiated into two primary tetrahedral lattices (PTL^+ and PTL^-); the geometry of each of these corresponds to that of the well-known octet truss popularized by R. Buckminster Fuller ([VT×OH], appearing as TH and OH. Each cube of a PCL can be differentiated into two tetrahedra as in the Stella Octangular, each belonging to one of the 2 PTLs of that PCL. Simply put, the nodes of the orthogonal axes of the edges of the cube of the PCL alternate to provide the nodes of its two PTLs.

Primary Polytopes (PPs) (of Class II of the Polyhedra) of the honeycombs are then disposed concentric with the nodes of the PCLs, and inherently of the PTLs. Three classes of honeycombs are identified. Firstly, in Class III of the Honeycombs, one PCL has PP_1 at its nodes, while the other PCL has PP_2 at its nodes (with or without intermediary (neutral) polyhedra). This means that the two PTLs of one PCL have the same PP at their nodes, while the other two have a different PP at their nodes, and their NTs, virtual or real, then differ. In some cases, PP_1 = PP_2, so both PCLs, and hence all 4 PTLs, have the same PP at each node; with or without intermediary (neutral) polyhedra, and their NTs are the same. Secondly, in Class II of the Honeycombs, the two PTLs of one PCL have different PPs (of Class II of the Polyhedra); the two PTLs of the other PCL then have either +ve and −ve TH at their

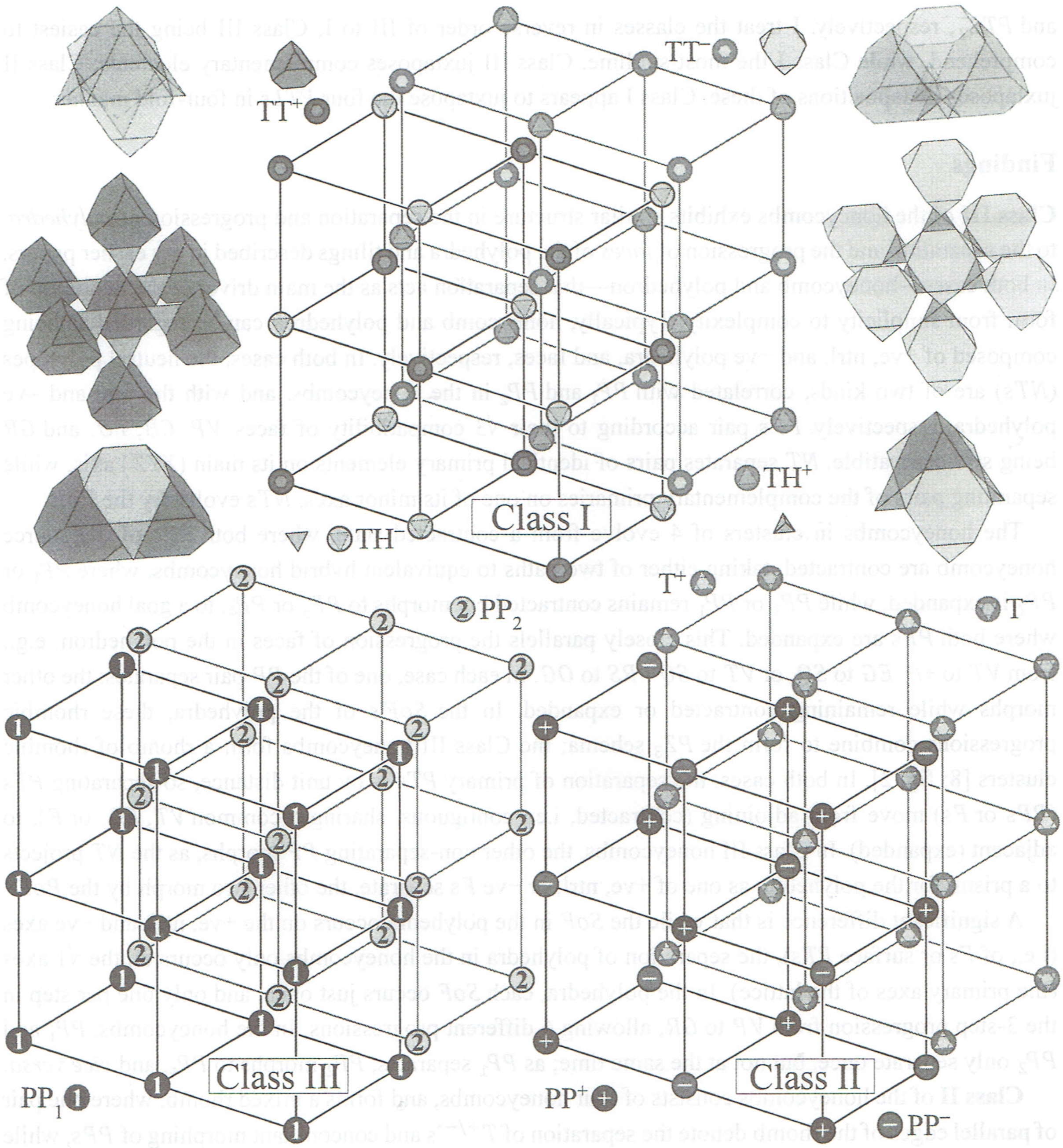

Fig. 1: Morphology of the three Classes of Honeycombs. Top: Class I of the Honeycombs. Center panel shows PTL_1^+ and PTL_1^- of PCL_1 alternate TT^+ and TH^- PPs, interspersed with PTL_2^+ and PTL_2^- of PCL_2 alternating TT^- and TH^+ PPs. Outliers show close packing of (irregular) octahedral cluster of square horizontal array and center vertical PPs above and below with (50 and 100) % opacity (top and bottom); orthogonal sequences on XYZ axes from a core TT (middle); similar arrangements could be drawn centered on either TH^+ or TH^-. Lower Right: Class II of the Honeycombs showing PTL_1^+ and PTL_1^- of PCL_1 alternate PPs of Class II of the Polyhedra (VP; OH, CO, CB; TO, SR, TC; GR), interspersed with PTL_2^+ and PTL_2^- of PCL_2, which alternate with $T^{+|-}$s, i.e., +ve & −ve THs, or +ve & −ve TTs. Lower Left: Class III of the Honeycombs showing PCL_1 consists of one PP of Class II of the Polyhedra, interspersed with PCL_2 also a PP of Class II. In some cases, the PPs of PCL_1 and PCL_2 are the same, e.g., the cubic array (contracted [CB×VP] or expanded [CB×CB]), [TO×TO], colored by PCL.

alternating nodes, or have +ve and −ve TT, respectively; these do not have neutral intermediary polyhedra, but are considered to have virtual NTs of VT, EG, or F of adjoining PPs or adjoining Ts (TH or TT). Thirdly, in Class I of the honeycombs, all 4 PTLs have different polyhedra at their nodes, and only one case appears to exist, of TH^+ and TT^- for PTL_1^+ and PTL_1^-, and TT^+ and TH^- for PTL_2^+

and PTL_2^-, respectively. I treat the classes in reverse order of III to I, Class III being the easiest to comprehend, while Class I the most sublime. Class III juxtaposes complementary elements; Class II juxtaposes juxtapositions of these; Class I appears to juxtapose the four PTLs in four-fold manner.

Findings

Class III of the honeycombs exhibits similar structure in the separation and progression of *polyhedra*, to the separation and the progression of *faces* of the polyhedra and tilings described in my earlier papers. In both cases—honeycomb and polyhedron—that separation acts as the main driver of the evolution of form from simplicity to complexity. Typically, honeycomb and polyhedron can be regarded as being composed of +ve, ntrl, and −ve polyhedra, and faces, respectively. In both cases, the neutral polytopes (NTs) are of two kinds, correlated with PP_1 and PP_2 in the honeycombs, and with the +ve and –ve polyhedra, respectively. PPs pair according to their √3 compatibility of faces, VP, CB, TO, and GR being self-compatible. NT separates pairs of identical primary elements on its main (XYZ) axis, while separating pairs of the complementary primaries on one of its minor axes. NTs evolve by the PoF.

The honeycombs in clusters of 4 evolve from a contracted state, where both PPs of the source honeycomb are contracted, taking either of two paths to equivalent hybrid honeycombs, where PP_1 or PP_2 is expanded, while PP_2 or PP_1 remains contracted but morphs to PP_3 or PP_4, to a goal honeycomb where both PPs are expanded. This closely parallels the progression of faces in the polyhedron, e.g., from VT to +/− EG to SQ, or VT to SQ / RS to OG. In each case, one of the PP pair separates; the other morphs while remaining contracted or expanded. In the SoFs of the polyhedra, these rhombic progressions combine to form the PZ_3 schema; the Class III honeycombs form a rhomb of rhombic clusters [8: fig. 2]. In both cases, the separation of primary PTs is by unit distance, so separating PTs (PPs or Fs) move from adjoining (contracted, i.e., contiguous, sharing a common VT, EG, or F), to adjacent (expanded). In Class III honeycombs, the other non-separating PP morphs, as the NT projects to a prism; for the polyhedra, as one of +ve, ntrl, or −ve Fs separate, the other two morph by the PoF.

A significant difference is that while the SoF in the polyhedra occurs on the +ve, ntrl, and −ve axes (i.e., of Fs or surface PTs), the separation of polyhedra in the honeycombs only occurs on the √1 axes (the primary axes of the lattice). In the polyhedra, each SoF occurs just once, and only one per step in the 3-step progression from VP to GR, allowing 6 different progressions. In the honeycombs, PP_1 and PP_2 only separate once, but not at the same time; as PP_1 separates, PP_2 morphs to PP_4, and *vice versa*.

Class II of the honeycombs consists of four honeycombs, and forms a mixed rhomb, where one pair of parallel edges of the rhomb denote the separation of $T^{+/-}$'s and concomitant morphing of PPs, while the other pair denote the truncation of $TH^{+/-}$'s to $TT^{+/-}$'s and concomitant morphing of PPs. PPs pair according to their √1 compatibility of faces: VT:OH, CB:SR, CO:TO, TC:GR. The +ve and −ve THs or TTs separate on the √1 axes, while PP_1 & PP_2 morph to PP_3 & PP_4. In both cases, the Ts separate from adjoining—sharing a NE, to adjacent—mediated by an EP on one of its minor axes. As this occurs, PP_1 and PP_2 evolve to PP_3 and PP_4, which are mediated by the same EP on its main (orthogonal) axis.

Class I with only one member would appear to allow no sequences.

Conclusion

Deeply akin to the SoF in the Polyhedra, the SoP in the Honeycombs appears to be the driving force in the development of spatial form from simple to complex. Virtual NTs that mediate adjoining PP_1s develop into regular prisms that separate adjacent PP_1s on √1 axes. In Class III, this formal development permeates the honeycombs; the SoP of one of the two PPs is always accompanied by a morphing of its √3 complementary PP, which develops according to a PoP, akin to the PoF in the polyhedra.

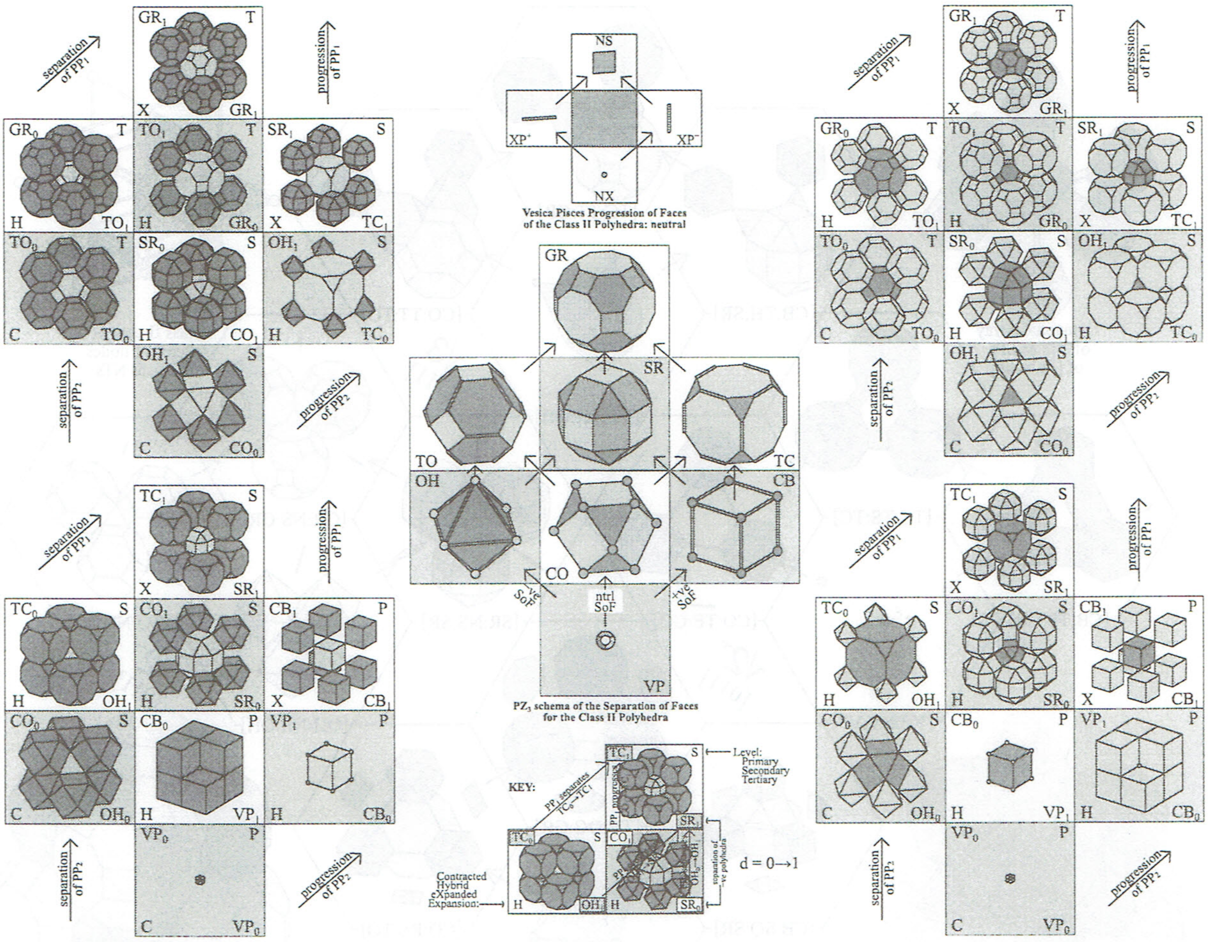

Fig. 2: Class III SoPs and PoPs of the PPs of the honeycombs, organized according to the SoFs and PoFs of the Class II Polyhedra. Top central schema shows the PoFs of the Class II Polyhedra, while central cluster of 8 PPs shows the SoFs of the Class II Polyhedra in the cubic PZ_3 schema consisting of lower and upper rhombs; −ve, ntrl, and +ve faces separate in rising left, vertically, and right, respectively (arrowed). These schema are applied to the PPs of the honeycombs in the four corner clusters of 8 honeycombs, respectively. Key shows that for each square, top left/ bottom right text of the four corner clusters identify PP_1/PP_2, respectively, hence total 16 (Class III) honeycombs $[PP_1 \times PP_2]$, in the left/right pair of clusters, respectively (disallowing reflections, 10 honeycombs); honeycombs correspond left/right, but are centered on −ve/+ve PP (PP_2/PP_1), respectively. In each square, top right text shows Primary, Secondary, or Tertiary (P,S,T) level of honeycomb; bottom right text shows Contracted, Hybrid, or Expanded (C,H,X) status of honeycomb. Closest outer PP of the outer 8 PPs of each honeycomb is omitted for clarity, as are the 3D NTs, while reinforcing the importance of the √3 axial $PP_1 \leftrightarrow PP_2$ matings. Grey squares show the lower rhomb of the SoF; bordered white squares show the upper rhomb. SoP of PP_2 and PoP of PP_1 are shown vertically from grey square to white square immediately above it, from d = 0 to 1, while SoP of PP_1 and PoP of PP_2 are shown from square to above right square, as per corner arrows. Left/right clusters are shown centered on the −ve/+ve cyan/magenta progressing $PP_{1/2}$, respectively. Each corner cluster of 8 rectangles shows the SoP of PP_2 and PoP of PP_1, or *vice versa*.

In Class II, two sequences of the SoP can be found as either $TH^{+/-}$'s or $TT^{+/-}$'s separate on √1 axes, while √2 complementary PP_1 and PP_2 both morph into more complex √1 complementary PP_3s and PP_4s. It would appear that Class I of only 1 honeycomb cannot allow sequences without breaking the constraint of all-space-filling periodicity. These regularities are addressed in greater detail in the companion paper that examines Class III of the Honeycombs [8], to be published in the December 2025 issue of Information journal, and available to download as a color PDF from Information or my website; and in intended subsequent papers. The morphologies are easier to appreciate from the illustrations.

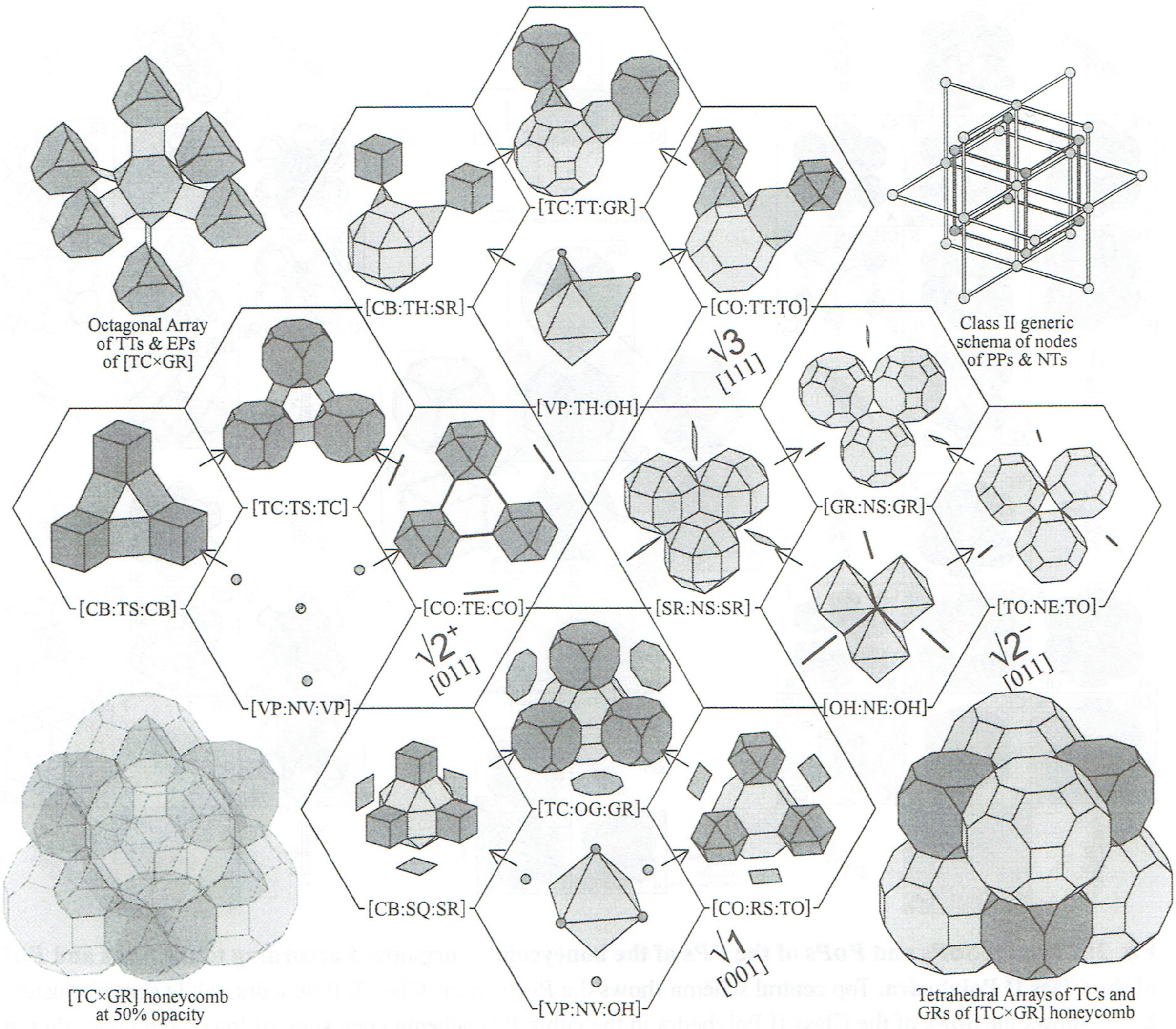

Fig. 3. Pairings of Class II polyhedra by √1, √2, √3 axes in the Class II honeycombs $[VP{\times}OH]$, $[CB{\times}SR]$, $[CO{\times}TO]$, $[TC{\times}GR]$ at bottom, left, right, and top, respectively, of each rhomb of 4 hexagons, with NT named and indicated between PP_1 and PP_2, and exploded when virtual. Schema of Class II nodes, top right. $[TC{\times}GR]$ honeycomb $PTLs$ (PPs, bottom right; TTs and EPs, top left); combined at 50 % opacity, bottom left.

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